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ANALYSIS OF HEAT OR MASS TRANSFER IN SOME COUNTERCURRENT FLOWS

RICHARD J. NUNGE and WILLIAM N. GILL
Department of Chemical Engineering and Metallurgy, Syracuse University, Syracuse, New York

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Abstract—An orthogonal expansion technique for solving some heat- or mass-transfer problems in counterflow systems is developed and applied to two idealized transport problems. The method of solution is based on Sturmian theory and requires both positive and negative eigenvalues for completeness. Local and mean Nusselt numbers are reported as functions of the parameters of the two systems.

An iterative finite difference method for solving the same problems is described and the results of the two methods compared.

NOMENCLATURE

$a/2$, distance between outer parallel plates;
B_n, expansion coefficients;
D, diffusivity;
d, distance between outer parallel plates in Fig. 1;
g, acceleration of gravity;
h, static head difference;
K, mass- or heat-transfer coefficient;
L, dimensionless length, $[l/(a/2)]/Pe$ for ice washing, or $[l/(a/4)]/Pe$ for heat exchanger;
l, length;
N, dimensionless constant, $[\rho g h(d/2)^2]/(2\mu l v_b)$;
Nu, Nusselt number;
Nu_m, mean Nusselt number defined by equation (34);
Pe, Péclet number;
r, radial coordinate;
r_i, position of the plane of zero velocity in Fig. 1;
T, temperature or concentration;
T_0, temperature or concentration at $z = 0$ and $r_i \leqslant r \leqslant d$;
T_l, temperature or concentration at $z = l$ and $0 \leqslant r \leqslant r_i$;
u, dimensionless velocity, v/v_b for ice washing, or v/v_{max} for heat exchanger;
v, velocity;
v_b, plate velocity for ice washing;
v_{max}, maximum velocity of slower stream for heat exchanger;
w/b, ratio of wash water to brine rates;
x, dimensionless axial coordinate, $[z/(a/2)]/Pe$ for ice washing, or $[z/(a/4)]/Pe$ for heat exchanger;
y, dimensionless radial coordinate, $r/(a/2)$ for ice washing, or $r/(a/4)$ for heat exchanger;
z, axial coordinate.

Greek symbols
Δ, dimensionless interfacial position, r_i/d;
θ, dimensionless concentration or temperature, $(T - T_0)/(T_l - T_0)$;
λ_n, eigenvalue;
μ, viscosity;
ρ, fluid density.

INTRODUCTION

IN RECENT years a wide variety of heat- and mass-transfer processes in bounded conduit flows have been successfully reduced to Sturm–Liouville systems and studied as such. Information gained from this work has markedly increased understanding of transport processes in systems where the axial velocity is unidirectional. Countercurrent processes, however, are fundamentally different since the velocity must change sign. Also, such problems are neither fully open-ended nor completely bounded in the conventional sense. This paper presents a method of solving the countercurrent problem in its simplest form; extention to a wider range of practical problems is possible and will be the subject of later papers.

Rigorous mathematical proofs of the existence of solutions to counterflow problems, or of the convergence of the type of orthogonal function expansions used in the present study to solve them, are not available. Consequently, in order to test the validity of the method developed here, and to extend the results to cases where the expansion is inconvenient, a finite difference iterative procedure which is a modification of King's [4] recent work, was used. Although no fully satisfactory formal method of interation which would handle all of the problems was found, it was possible to establish very good agreement between the finite difference approach and the orthogonal function expansion for all cases investigated.

THEORETICAL DEVELOPMENT

The essential features of the general problem studied are shown on Fig. 1. It is seen that the velocity, $v(r)$, is an arbitrary continuous function of r which assumes both positive and negative values. Also, the dependent variable is specified at $z = l$ over the interval $0 \leqslant r \leqslant r_i$ and at $z = 0$ over the interval $r_i \leqslant r \leqslant d$. The interface at $r = r_i$ may be a solid boundary having no resistance or it may be just a plane of zero velocity, but in any event it is assumed to be a plane which effectively separates the flow into two streams. In one stream the velocity is positive everywhere and in the other it is negative everywhere with respect to a stationary frame of reference.

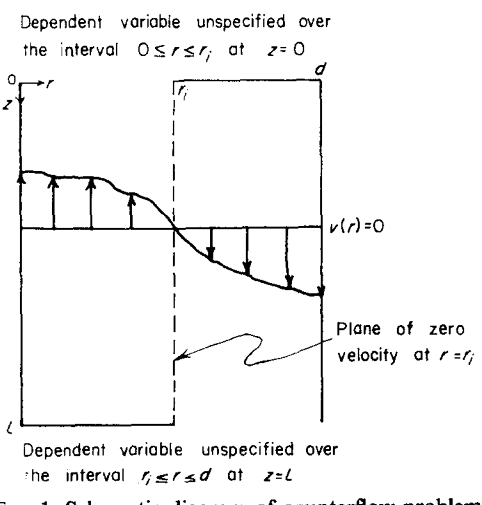

FIG. 1. Schematic diagram of counterflow problem.

The most important simplifying assumptions made in this analysis are:

1. The velocity field is fully developed in that it is a function of r only.
2. The flow is laminar.
3. Molecular diffusion or conduction in the axial direction is negligible compared with convection.
4. Physical properties are constant and the same in both streams.
5. There is no transport resistance or capacitance in the plane of zero velocity which separates the streams.

To reduce the number of parameters involved it is further stipulated that no heat or mass is transferred across the outer boundaries of the system at $r = 0$ and $r = d$, and that each stream enters with a uniform temperature or concentration profile.

A few comments regarding assumptions four and five seem in order. First, it is precisely when the properties of two countercurrent streams are similar that we know least about them since the idea of a controlling resistance has no validity whatsoever. Secondly, in numerous heat- and mass-transfer applications, but by no means in all cases, the interfacial resistance can be considered negligible. Therefore, it seems quite reasonable to make these assumptions in a first attempt at solving analytically some counterflow transport processes.

With the dimensionless axial and radial distances

$$x = \frac{z}{Pe\,.\,d}$$

$$y = r/d$$

the entire temperature or concentration field is described by the following system of equations

$$u(y)\frac{\partial\theta}{\partial x}=\frac{\partial^2\theta}{\partial y^2};\ 0\leqslant y\leqslant 1;\ 0\leqslant x\leqslant L \quad (1)$$

$$\frac{\partial\theta}{\partial y}(x,0)=0 \quad (2)$$

$$\frac{\partial\theta}{\partial y}(x,1)=0 \quad (3)$$

$$\theta(0,y)=0;\ \Delta\leqslant y\leqslant 1 \quad (4)$$

$$\theta(L,y)=1;\ 0\leqslant y\leqslant \Delta \quad (5)$$

where θ is the dimensionless concentration or temperature, $\theta = (T - T_0)/(T_l - T_0)$, and $\Delta = r_i/d$. Using separation of variables leads to

$$X_n(x)=\exp[-\lambda_n x]$$

$$Y_n''(y)+\lambda_n u(y)\,Y_n(y)=0;\ 0\leqslant y\leqslant 1 \quad (6)$$

$$Y_n'(0)=0 \quad (7)$$

$$Y_n'(1)=0 \quad (8)$$

This is a special case of the Sturm problem for which it has been shown [3] that when $u(y)$ changes sign over the interval in question that both a positive and a negative set of real eigenvalues which have the limit points $+\infty$ and $-\infty$ respectively, exist. Furthermore, with boundary conditions given by equations (7) and (8), Bocher [2] has shown that the smallest eigenvalue in absolute value of either or both sets will be zero depending on the value of F, where

$$F=\int_0^1 u(y)\,dy$$

If $F = 0$, $\lambda_0^+ = \lambda_0^- = 0$, if $F > 0$, $\lambda_0^+ = 0$, and if $F < 0$, $\lambda_0^- = 0$. Thus, for counterflow systems it is necessary to find a convergent series in both positive and negative eigenvalues, since each set must be retained if the orthogonal function expansion is to be complete. The solution is of the form

$$\theta(x,y)=\sum_{n=0}^{\infty} B_n^+\,X_n^+(x)\,Y_n^+(y)+\sum_{n=0}^{\infty} B_n^-\,X_n^-(x)\,Y_n^-(y) \quad (9)$$

which will be written here as simply

$$\theta(x,y)=\sum_{n=0}^{\infty} B_n\,X_n(x)\,Y_n(y) \quad (10)$$

To derive expressions for the expansion coefficients the entrance conditions, equations (4) and (5), are rewritten as

$$\theta(0,y)=\begin{cases} f(y); & 0\leqslant y\leqslant\Delta \\ 0; & \Delta\leqslant y\leqslant 1\end{cases} \quad (11)$$

$$\theta(L,y)=\begin{cases} 1; & 0\leqslant y\leqslant\Delta \\ p(y); & \Delta\leqslant y\leqslant 1\end{cases} \quad (12)$$

where

$$f(y)=\sum_{q=0}^{\infty} B_q\,Y_q(y);\ 0\leqslant y\leqslant\Delta \quad (13)$$

$$p(y)=\sum_{q=0}^{\infty} B_q\exp[-\lambda_q L]\,Y_q(y);\ \Delta\leqslant y\leqslant 1 \quad (14)$$

Equations (13) and (14) follow from the form of the solution given by equation (10). If one notes that the Y_n's are orthogonal with respect to $u(y)$ over (0, 1) and that

$$\theta(0,y)=\sum_{n=0}^{\infty} B_n\,Y_n(y) \quad (15)$$

$$\theta(L,y)=\sum_{n=0}^{\infty} B_n\exp[-\lambda_n L]\,Y_n(y) \quad (16)$$

then it can be seen that orthogonal expansions can be written at both ends of the model. Multiplying both sides of equation (15) by $u(y)\,Y_m(y)\,dy$ and integrating over (0, 1) yields

$$\int_0^1 u(y)\,\theta(0,y)\,Y_m(y)\,dy=\sum_{n=0}^{\infty} B_n\int_0^1 u(y)\,Y_n(y)\,Y_m(y)\,dy \quad (17)$$

As a consequence of the orthogonality conditions, the right-hand side of equation (17) is zero unless $m = n$. Therefore

$$B_n=\frac{\int_0^1 u(y)\,\theta(0,y)\,Y_n(y)\,dy}{\int_0^1 u(y)\,[Y_n(y)]^2\,dy} \quad (18)$$

Similarly at $x = L$,

$$B_n = \exp[\lambda_n L] \frac{\int_0^1 u(y)\, \theta(L, y)\, Y_n(y)\, dy}{\int_0^1 u(y)\, [Y_n(y)]^2\, dy} \tag{19}$$

But since $\theta(0, y) = 0$, $\Delta \leqslant y \leqslant 1$,

$$\int_0^1 u(y)\, \theta(0, y)\, Y_n(y)\, dy = \int_0^\Delta u(y) f(y)\, Y_n(y)\, dy$$

or, in view of equation (13)

$$\int_0^1 u(y)\, \theta(0, y)\, Y_n(y)\, dy = \sum_{q=0}^{\infty} B_q \int_0^\Delta u(y)\, Y_q(y)\, Y_n(y)\, dy \tag{20}$$

The integral in the numerator of equation (19) can also be rewritten by using equations (12) and (14) as

$$\int_0^1 u(y)\, \theta(L, y)\, Y_n(y)\, dy = \int_0^\Delta u(y)\, Y_n(y)\, dy + \sum_{q=0}^{\infty} B_q \exp[-\lambda_q L] \int_\Delta^1 u(y)\, Y_q(y)\, Y_n(y)\, dy \tag{21}$$

The right-hand side of equations (18) and (19) are equal and thus by equating these two expressions for B_n and substituting equations (20) and (21) in the result, after some simple rearrangement, one gets

$$\exp[\lambda_n L] \int_0^\Delta u(y)\, Y_n(y)\, dy = \sum_{q=0}^{\infty} B_q \left\{ \int_0^\Delta u(y)\, Y_q(y)\, Y_n(y)\, dy - \exp[(\lambda_n - \lambda_q) L] \int_\Delta^1 u(y)\, Y_n(y)\, Y_q(y)\, dy \right\} \tag{22}$$

Equation (22) applies for all values of n for both positive and negative sets of eigenvalues and therefore constitutes a set of linear equations, which defines the B_q. Thus, to find the expansion coefficients a set of simultaneous equations must be solved. Note that in this case each successive term added to the solution has an effect on the preceding expansion coefficients, which is not the case in the ordinary use of orthogonal functions such as in the Graetz problem.

Standard simplifying techniques using equations (6) to (8) can be employed to obtain expressions for the integrals in equation (22). Thus, for $\lambda_n \neq 0$, equation (22) becomes

$$-\frac{\exp[\lambda_n L]}{\lambda_n} Y_n'(\Delta) = \sum_{q=0}^{\infty} B_q \begin{cases} \dfrac{1}{\lambda_q - \lambda_n} [Y_n'(\Delta)\, Y_q(\Delta) - Y_q'(\Delta)\, Y_n(\Delta)]\, \{1 + \exp[(\lambda_n - \lambda_q) L]\}; & q \neq n \\ 2 Y_n'(\Delta) \dfrac{\partial Y_n}{\partial \lambda_n}(\Delta) - 2 Y_n(\Delta) \dfrac{\partial Y_n'}{\partial \lambda_n}(\Delta) + Y_n(1) \dfrac{\partial Y_n'(1)}{\partial \lambda_n}; & q = n \end{cases} \tag{23}$$

where the primes refer to differentiation with respect to y. For $\lambda_n^+ = 0$

$$\int_0^\Delta u(y)\, dy = B_0^+ \left\{ \int_0^\Delta u(y)\, dy - \int_\Delta^1 u(y)\, dy \right\} + \sum_{q=1}^{\infty} B_q^+ \frac{Y_q^{+\prime}(\Delta)}{\lambda_q^+} [1 + \exp(-\lambda_q^+ L)] - \sum_{q=0}^{\infty} B_q^- \frac{Y_q^-}{\lambda_q^-} [1 + \exp(-\lambda_q^- L)] \tag{24}$$

METHOD OF SOLUTION

To deal with any velocity profile, which is an arbitrary function of y, a Runge–Kutta numerical integration of equation (6) was used. Without loss of generality it can be assumed that $Y_n(0) = 1$. This condition along with equation (7) and an initial guess of the eigenvalue form a starting point for the integration. Repeated integrations with adjustments of the eigenvalue by a false position method were made until equation (8) was satisfied. Values of the derivatives with respect to λ_n in equation (23) were also found by this method after differentiating equation (6) with respect to λ_n. A Gauss–Jordan reduction was used to solve the set of simultaneous equations generated by equations (23) and (24). All calculations were made on an IBM 7070 computer.

PHYSICAL PROBLEMS

The mathematical problem and the method for solving it, which were discussed previously, are sufficiently general to be useful in analysing a number of rather interesting physical problems. The most obvious application of the foregoing analysis is the counterflow heat exchanger, but before considering this, a simplified version of an interesting mass-transfer problem which arises in the freezing process for demineralizing sea water shall be discussed.

An important step in the freezing process is the steady-state countercurrent washing of ice particles with fresh water to remove brine as described by Barduhn [1]. This occurs in a vertical column equipped to feed a slurry of ice and brine at the bottom and fresh water at the top. As an analogy to the flow in porous media which occurs in the freezing process, consider instead that flow takes place between infinitely wide moving flat plates, with a distance between plates on the order of an ice particle diameter, a. Each plate might be considered part of a continuous belt which moves on rollers at a constant velocity and the bottom of the two belt systems, which constitute the two moving plates, is immersed in a tank in which the mean brine concentration is maintained constant. Brine adheres to the plate surfaces as they move out of the brine solution. The velocity of the plates, v_b, is representative of the ice bed velocity up the column. The space between plates is completely filled with fluid and part of the fluid moves upward with the plates. Fluid around the centerline moves downward but is replaced continuously at the top so that the space between plates is always full. A pressure gradient may exist in the fluid which will be measured by a static head difference, h. Flow is laminar and symmetric about the centerline and only one-half of the model need be considered. Thus, the velocity distribution is given by

$$u(y) = 1 - N(1 - y^2) \qquad (25)$$

where

$$y = \frac{r}{a/2}$$

$$N = \frac{\rho g h\,(a/2)^2}{2\mu l v_b}$$

and equation (1) describes the mass transfer in this system.

The value of N determines the position of the interface between the two streams. That is, the point of zero velocity occurs at

$$\Delta = [1 - (1/N)]^{1/2}$$

The net volumetric flow rate is represented by F, where

$$F = \int_0^1 u(y)\,\mathrm{d}y$$

In practical applications, F is close to zero (if $F = 0$, $N = 1{\cdot}5$, and $\Delta = 0{\cdot}577$), with $F < 0$ indicating a net flow down the column. Table 1 shows the rates which have been investigated with w/b defined as the ratio of wash water to brine rate.

Table 1. Flow rate parameters

N	Δ	F	w/b
1·73	0·65	−0·155	1·95
1·33	0·5	0·111	0·5
1·14	0·35	0·24	0·111

Table 2. Parameters for ice washing calculations

n	λ_n^+	$Y_n^{+\prime}(\Delta)$	B_n^+			
			$L = 0{\cdot}005$	$L = 0{\cdot}01$	$L = 0{\cdot}05$	$L = 0{\cdot}1$
			$N = 1{\cdot}73$			
0	6·04660	2·26807	−0·855100	−0·821946	−0·670164	−0·618004
1	304·479	$1{\cdot}04484 \times 10^{4}$	$-7{\cdot}52734 \times 10^{-5}$	$-9{\cdot}84213 \times 10^{-5}$	$-4{\cdot}68082 \times 10^{-5}$	$-1{\cdot}25824 \times 10^{-5}$
2	998·365	$8{\cdot}33279 \times 10^{5}$	$-1{\cdot}99370 \times 10^{-7}$	$1{\cdot}23264 \times 10^{-8}$	$-7{\cdot}63846 \times 10^{-8}$	—
3	—	—	—	—	—	—
			$N = 1{\cdot}33$			
0	0	0	−0·884548	−0·808050	−0·452056	−0·229514
1	156·862	48·2386	$-1{\cdot}04459 \times 10^{-2}$	$-9{\cdot}38268 \times 10^{-3}$	$-2{\cdot}4023 \times 10^{-3}$	$-2{\cdot}6293 \times 10^{-3}$
2	516·639	$8{\cdot}25016 \times 10^{2}$	$-9{\cdot}37561 \times 10^{-4}$	$-7{\cdot}64941 \times 10^{-4}$	$-7{\cdot}2229 \times 10^{-4}$	$-3{\cdot}3810 \times 10^{-4}$
			$N = 1{\cdot}14$			
0	0	0	$-9{\cdot}1546 \times 10^{-2}$	$-6{\cdot}9905 \times 10^{-2}$	$-5{\cdot}2620 - 10^{-3}$	$-6{\cdot}82 \times 10^{-4}$
1	95·2042	3·59257	$-5{\cdot}45833 \times 10^{-2}$	$-4{\cdot}57064 \times 10^{-2}$	$5{\cdot}11851 \times 10^{-4}$	—
2	326·970	$1{\cdot}71501 \times 10^{2}$	$-7{\cdot}77130 \times 10^{-3}$	$-2{\cdot}27128 \times 10^{-2}$	$-2{\cdot}81213 \times 10^{-4}$	—
3	688·178	$5{\cdot}67898 \times 10^{2}$	$-1{\cdot}21904 \times 10^{-2}$	—	—	—

n	λ_n^-	$Y_n^{-\prime}(\Delta)$	B_n^-			
			$L = 0{\cdot}005$	$L = 0{\cdot}01$	$L = 0{\cdot}05$	$L = 0{\cdot}1$
			$N = 1{\cdot}73$			
0	0	0	1·95531	1·88964	1·56650	1·39953
1	−80·4435	3·09640	−0·121791	$-7{\cdot}48072 \times 10^{-2}$	$-1{\cdot}15911 \times 10^{-3}$	$-3{\cdot}84489 \times 10^{-5}$
2	−262·020	−5·07234	$2{\cdot}73452 \times 10^{-2}$	$6{\cdot}96547 \times 10^{-3}$	$1{\cdot}18377 \times 10^{-7}$	—
3	−546·744	6·85338	$-8{\cdot}42295 \times 10^{-3}$	$-4{\cdot}43278 \times 10^{-4}$	$-1{\cdot}50988 \times 10^{-13}$	—
			$N = 1{\cdot}33$			
0	−8·68774	−0·901755	1·92184	1·75977	0·990162	0·531144
1	−299·998	4·10413	$-4{\cdot}01187 \times 10^{-2}$	$-1{\cdot}07868 \times 10^{-2}$	$-3{\cdot}42174 \times 10^{-8}$	$-1{\cdot}73557 \times 10^{-14}$
2	−971·999	−6·59874	$1{\cdot}61412 \times 10^{-3}$	$1{\cdot}55746 \times 10^{-6}$	$8{\cdot}03285 \times 10^{-23}$	$-1{\cdot}2100 \times 10^{-44}$
			$N = 1{\cdot}14$			
0	−58·2401	−1·72662	0·943226	$6{\cdot}99333 \times 10^{-2}$	$5{\cdot}82915 \times 10^{-2}$	$3{\cdot}2587 \times 10^{-3}$
1	−1461·89	1·91323	$-4{\cdot}50436 \times 10^{-4}$	$-1{\cdot}15837 \times 10^{-7}$	$-1{\cdot}20880 \times 10^{-32}$	—
2	—	—	—	—	—	—
3	—	—	—	—	—	—

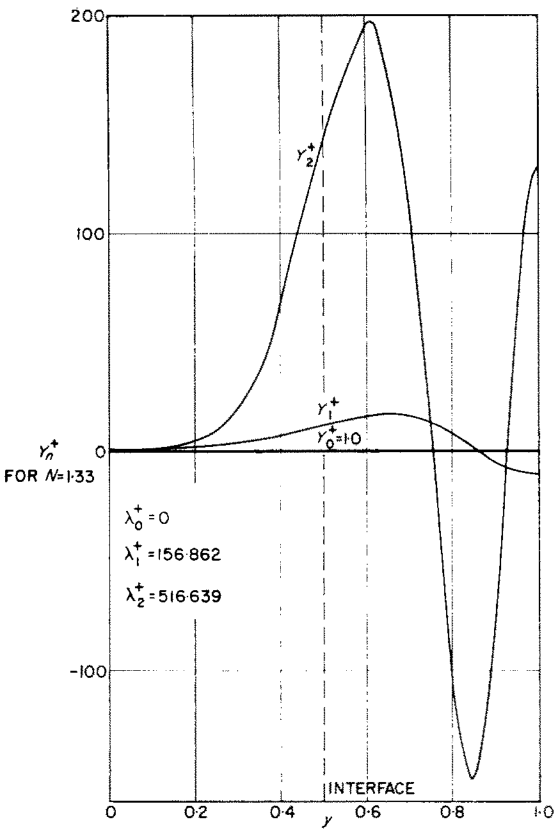

FIG. 2. Positive eigenfunctions.

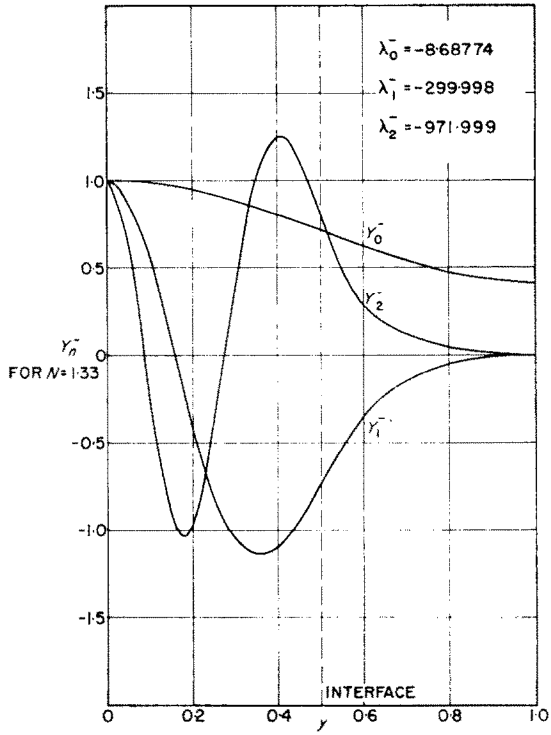

FIG. 3. Negative eigenfunctions.

Several eigenvalues and eigenfunctions have been calculated for each value of N listed in Table 1. These eigenvalues and the other parameters necessary to calculate transport coefficients are tabulated in Table 2. Figures 2 and 3, for $N = 1 \cdot 33$, illustrate typical behavior of the eigenfunctions. Both the positive and negative sets have characteristics which hinder the calculation of higher order functions. Notice the tremendous variations in magnitude of the higher order positive functions and note also that the higher order negative functions approach zero closer and closer to the interface. It is also worth noting that eigenfunctions corresponding to the positive set of eigenvalues are periodic in the region of positive velocity and eigenfunctions of the negative set are periodic in the region of negative velocity as required by Sturm's comparison theorems. Furthermore, this seems to be intimately related to the approximation of the entrance conditions. The addition of more negative terms to the series improves the fit in the region of negative velocity while the addition of more positive terms to the series has a greater effect in the region of positive velocity.

Figure 4 illustrates the improvement of entrance conditions obtained by the addition of more terms to the series. As in all problems solved with orthogonal function expansions an accurate description of the region very close to the entrance, or in this case at both ends of the model, requires a large number of terms in the series. It is for this reason that asymptotic expressions for the large eigenvalues and corresponding eigenfunctions are very useful.

The primary interest in solving transport problems is in obtaining an indication of transfer rates. For this purpose a mass-transfer

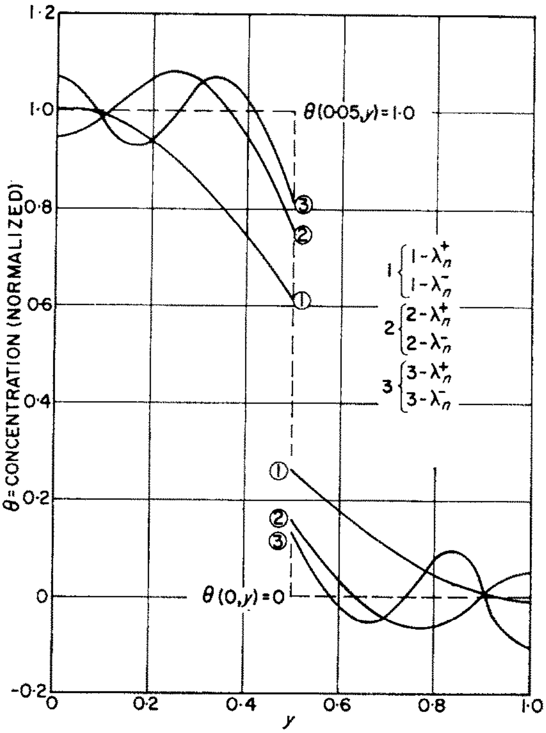

FIG. 4. Effect of increasing number of terms in eigenfunction expansion on fitting end conditions for the case $L = 0{\cdot}05$, $N = 1{\cdot}33$.

coefficient, K, is defined as

$$K(\Delta\theta_B) = -D\frac{\partial\theta}{\partial r}\bigg|_{r_i}$$

where $\Delta\theta_B$ is the difference in bulk concentrations of the two streams. The Nusselt number thus is given by

$$Nu(x) = \frac{K(x)\,a/2}{D} = -\frac{\partial\theta/\partial y\,|_\Delta}{\Delta\theta_B}$$

To derive an expression for the Nusselt number in terms of expansions consider $\lambda_0^- = 0$ (a similar procedure applies for $\lambda_0^- = 0$), and return to using the form of solution shown in equation (9). The difference in bulk concentrations, $\Delta\theta_B$ is given by

$$\Delta\theta_B = \frac{\int_\Delta^1 u(y)\,\theta(x,y)\,\mathrm{d}y}{\int_\Delta^1 u(y)\,\mathrm{d}y} - \frac{\int_0^\Delta u(y)\,\theta(x,y)\,\mathrm{d}y}{\int_0^\Delta u(y)\,\mathrm{d}y}$$

An expression for the integrals involving $\theta(x, y)$ above is now derived by substituting equation (9) for $\theta(x, y)$ and simplifying.

$$\int_\Delta^1 u(y)\,\theta(x,y)\,\mathrm{d}y = \sum_{n=0}^{\infty} B_n^+ X_n^+ \int_\Delta^1 u(y)\,Y_n^+(y)\,\mathrm{d}y + \sum_{n=0}^{\infty} B_n^- X_n^- \int_\Delta^1 u(y)\,Y_n^-\,\mathrm{d}y =$$

$$B_0^+ \int_\Delta^1 u(y)\,\mathrm{d}y + \sum_{n=1}^{\infty} B_n^+ X_n^+ \frac{Y_n^{+\prime}(\Delta)}{\lambda_n^+} + \sum_{n=0}^{\infty} B_n^- X_n^- \frac{Y_n^{-\prime}(\Delta)}{\lambda_n^-}$$

Thus,

$$\Delta\theta_B = B_0^+ + \frac{\sum_{n=1}^{\infty} B_n^+ X_n^+ [Y_n^{+\prime}(\Delta)/\lambda_n^+]}{\int_\Delta^1 u(y)\,\mathrm{d}y} + \frac{\sum_{n=0}^{\infty} B_n^- X_n^- [Y_n^{-\prime}(\Delta)/\lambda_n^-]}{\int_\Delta^1 u(y)\,\mathrm{d}y}$$

$$- B_0^+ + \frac{\sum_{n=1}^{\infty} B_n^+ X_n^+ [Y_n^{+\prime}(\Delta)/\lambda_n^+]}{\int_0^\Delta u(y)\,\mathrm{d}y} + \frac{\sum_{n=0}^{\infty} B_n^- X_n^- [Y_n^{-\prime}(\Delta)/\lambda_n^-]}{\int_0^\Delta u(y)\,\mathrm{d}y}$$

$$= \frac{[\int_0^1 u(y)\,\mathrm{d}y]\{\sum_{n=1}^{\infty} B_n^+ X_n^+ [Y_n^{+\prime}(\Delta)/\lambda_n^+] + \sum_{n=0}^{\infty} B_n^- X_n^- [Y_n^{-\prime}(\Delta)/\lambda_n^-]\}}{[\int_0^\Delta u(y)\,\mathrm{d}y]\,[\int_\Delta^1 u(y)\,\mathrm{d}y]}$$

Therefore, since $Y_0^{+\prime}(\Delta)$ is zero, the Nusselt number is given by

$$Nu(x) = -\frac{[\int_0^{\Delta} u(y)\,dy]\,[\int_{\Delta}^{1} u(y)\,dy]}{\int_0^1 u(y)\,dy}\left(\frac{\sum_{n=1}^{\infty} B_n^+ X_n\, Y_n^{+\prime}(\Delta) + \sum_{n=0}^{\infty} B_n^- X_n^-\, Y_n^{-\prime}(\Delta)}{\sum_{n=1}^{\infty} B_n^+ X_n^+\,[Y_n^{+\prime}(\Delta)/\lambda_n^+] + \sum_{n=0}^{\infty} B_n^- X_n^-\,[Y_n^{-\prime}(\Delta)/\lambda_n^-]}\right)$$

The variation of the Nusselt number with x is given in Table 3. Notice that the values given for $x = 0$ and $x = L$ are not expected to be accurate.

Table 3. Nusselt number variation with N, L and x

N	L	Nusselt number					
		$\frac{x}{L} = 0$	$\frac{x}{L} = 0{\cdot}2$	$\frac{x}{L} = 0{\cdot}4$	$\frac{x}{L} = 0{\cdot}6$	$\frac{x}{L} = 0{\cdot}8$	$\frac{x}{L} = 1{\cdot}0$
1·73	0·005	5·18	4·03	3·66	3·67	3·93	4·47
1·73	0·01	3·31	3·01	2·88	2·96	3·30	4·19
1·73	0·05	3·63	2·06	2·09	2·18	2·43	3·16
1·73	0·1	2·20	2·02	2·02	2·05	2·21	3·11
1·33	0·005	3·92	3·25	3·04	3·11	3·39	4·0
1·33	0·01	4·10	2·46	2·27	2·27	2·42	4·38
1·33	0·05	3·53	1·99	1·94	1·93	1·96	3·47
1·33	0·1	3·66	1·95	1·93	1·93	1·93	2·94
1·14	0·005	6·75	3·78	2·91	2·65	2·75	3·61
1·14	0·01	5·75	3·14	2·56	2·38	2·33	2·74
1·14	0·05	6·31	2·15	2·15	2·15	2·15	3·55
1·14	0·1	2·15	2·15	2·15	2·15	2·15	2·15

Table 4. Parameters for countercurrent heat exchanger

n	$\lambda_n{}^+$	$Y_n{}^{+\prime}(\Delta)$	$B_n{}^+$		
			$L = 0{\cdot}01$	$L = 0{\cdot}05$	$L = 0{\cdot}1$
0	0	0	−4·44005	−3·33931	−2·61323
1	89·9397	68·2672	$-5{\cdot}5637 \times 10^{-3}$	$-2{\cdot}45692 \times 10^{-3}$	$-4{\cdot}27125 \times 10^{-3}$
2	284·222	$1{\cdot}90287 \times 10^{3}$	$-2{\cdot}63944 \times 10^{-4}$	$-2{\cdot}91375 \times 10^{-4}$	—

n	$\lambda_n{}^-$	$Y_n{}^{-\prime}(\Delta)$	$B_n{}^-$		
			$L = 0{\cdot}01$	$L = 0{\cdot}05$	$L = 0{\cdot}1$
0	−1·34714	−0·218101	5·50652	4·15168	3·24034
1	−110·158	2·02474	$-6{\cdot}83725 \times 10^{-2}$	$-4{\cdot}22447 \times 10^{-4}$	$-2{\cdot}88601 \times 10^{-6}$
2	−344·721	−2·98728	$-8{\cdot}43787 \times 10^{-4}$	$-7{\cdot}00989 \times 10^{-9}$	—

One of the most interesting and useful heat transfer devices is the counterflow heat exchanger which has been employed for many years in industrial applications, but has not yet been successfully analysed theoretically. The present method enables one to analyse at least a simple version of the counterflow heat exchanger. It is assumed that the system consists of three plates, that the outer two are perfectly insulated, and that the inner plate has no resistance. The same fluid is in laminar flow on both sides of the inner plate but the flows are in opposite directions. This problem has been solved for equal-sized streams with different maximum velocities. If the width of each stream is $a/4$ and y is the dimensionless distance, $r/(a/4)$, measured from the lower plate, then the velocity profile used is given by

$$u(y) = \frac{v}{v_{max}} = \begin{cases} y(y-1) & ; 0 \leqslant y \leqslant 1 \\ 1{\cdot}2\,(y-1)(2-y) & ; 1 \leqslant y \leqslant 2 \end{cases}$$

where v_{max} is the maximum velocity of the slower stream. Table 4 lists the parameters necessary to calculate the transfer coefficients and Table 5 gives the Nusselt number variation for this system.

Table 5. *Nusselt number variation with L and x*

$L = \frac{l/a/4}{Pe}$	Nusselt number $= K(a/4)/D$					
	$x/L = 0$	$x/L = 0{\cdot}2$	$x/L = 0{\cdot}4$	$x/L = 0{\cdot}6$	$x/L = 0{\cdot}8$	$x/L = 1{\cdot}0$
0·01	1·70	1·68	1·67	1·67	1·68	1·71
0·05	2·44	1·50	1·40	1·40	1·48	2·52
0·1	1·91	1·44	1·36	1·36	1·41	1·93

FINITE DIFFERENCE TECHNIQUE

Since the analytical method used in solving these problems has not been proven in a strict mathematical sense, and since King [4] has developed a method for solution of the infinite stream problem which requires a reasonable amount of computer time, his technique has been modified to solve the bounded system. For finite difference calculations it is necessary to

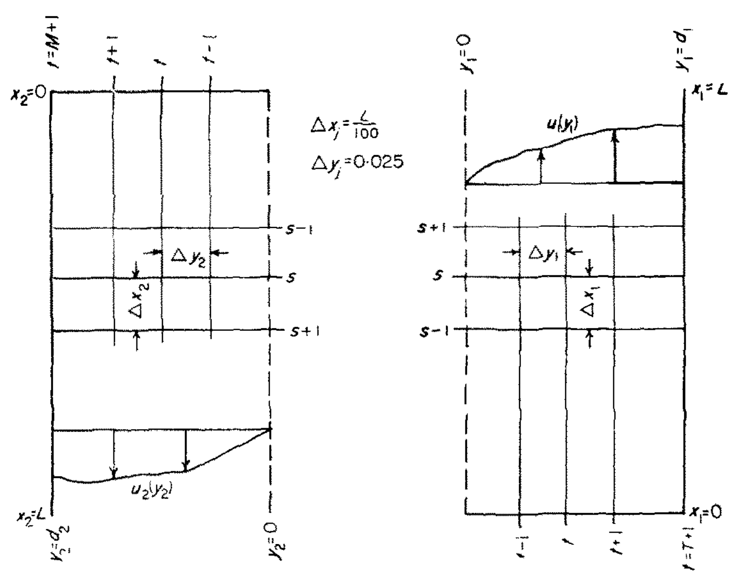

FIG. 5. Schematic diagram of finite difference model.

consider each stream separately as illustrated in Fig. 5. The equations to be solved for this form of the problem are

$$u_j(y_j)\frac{\partial \theta_j}{\partial x_j} = \frac{\partial^2 \theta_j}{\partial {y_j}^2}; \quad 0 \leqslant y_j \leqslant d_j; \quad 0 \leqslant x_j \leqslant L; \quad j = 1,2 \tag{26}$$

$$\theta_j(0, y_j) = \begin{cases} 0, j = 1 \\ 1, j = 2 \end{cases} \tag{27}$$

$$\frac{\partial \theta_j}{\partial y_j}(x_j, d_j) = 0; \quad j = 1,2 \tag{28}$$

$$\theta_1(x_1, 0) = \theta_2(L - x_1, 0) \tag{29}$$

$$\frac{\partial \theta_1}{\partial y_1}(x_1, 0) = -\frac{\partial \theta_2}{\partial y_2}(L - x_1, 0) \tag{30}$$

The two additional conditions, equations (29) and (30), evolved from breaking up the problem, are automatically satisfied in the series solution. Considering Fig. 5, equation (26) for either stream in terms of the Crank–Nicholson implicit finite difference form is

$$\theta_{s+1,\, t+1} - 2\left[1 + \frac{u_t(\Delta y)^2}{\Delta x}\right]\theta_{s+1,\, t} + \theta_{s+1,\, t-1} = 2\left[1 - \frac{u_t(\Delta y)^2}{\Delta x}\right]\theta_{s,\, t} - \theta_{s,\, t-1} - \theta_{s,\, t+1} \tag{31}$$

Equation (28) rewritten in difference form becomes

$$\frac{\theta_{s+1,\, T+2} - \theta_{s+1,\, T}}{2\Delta y_1} = 0$$

or

$$\theta_{s+1,\, T+2} = \theta_{s+1,\, T} \tag{32}$$

and similarly

$$\theta_{s+1,\, M+2} = \theta_{s+1,\, M} \tag{33}$$

To solve this model, first an initial guess of the interfacial distribution is made (or results of series calculations are used). With this prescribed, the distribution of θ in each phase or stream is found by starting at the end where the entrance value of θ is given and then solving (5) the sets of simultaneous equations generated by equations (30), (31) and (32) by the Thomas method. Since the same interfacial concentration is used for each stream, equation (29) is automatically satisfied. As suggested by King [4] the derivatives at $y = 0$ given by

$$\left.\frac{\partial \theta}{\partial y_j}\right|_0 = \frac{1}{12\Delta y_j}(-25\,\theta_{s,\,0} + 48\,\theta_{s,\,1} - 36\,\theta_{s,\,2} + 16\,\theta_{s,\,3} - 3\,\theta_{s,\,4})$$

are then checked to test the degree to which they satisfy equation (30). If they differ by more than a pre-assigned allowable error, the interfacial concentration or temperature at that point is adjusted and the concentration distribution recalculated. This procedure is repeated until all the derivatives agree to ± 8 per cent or less. (After convergence, an overall energy balance for one case checked within 3 per cent). It is difficult to give a general procedure for adjusting the interfacial θ to speed convergence, since all the problems solved by this technique had different requirements. Examination of intermediate results and experience with the method proved best. Another disadvantage of this method when compared to the series method is that for every combination of parameters the entire problem must be started over; thus, one uses a large amount of computer time if a wide range of parameters is investigated.

Several problems, however, have been solved using this method for comparison with series results. Excellent agreement was obtained as is evident from Figs. 6, 7 and 8. Furthermore, this method has been used to investigate very small lengths where the number of eigenvalues required in the analytical method becomes excessive. Thus, Fig. 9, which shows the variation of a mean Nusselt number defined as

$$Nu_m = \frac{1}{L}\int_0^L Nu(x)\,\mathrm{d}x \tag{34}$$

is a combination of results of both series and

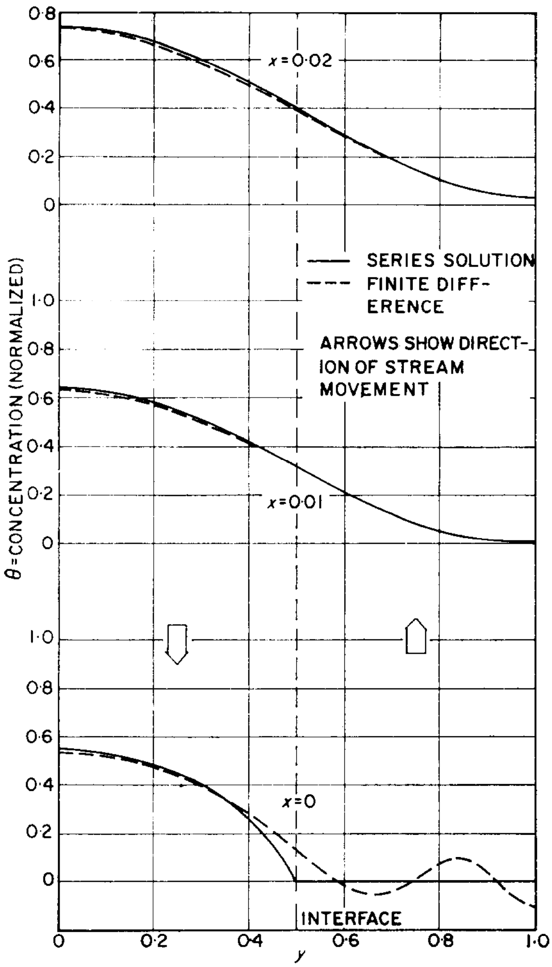

FIG. 6. Transverse dimensionless concentration distributions at various values of x for the case $L = 0{\cdot}05$, $N = 1{\cdot}33$.

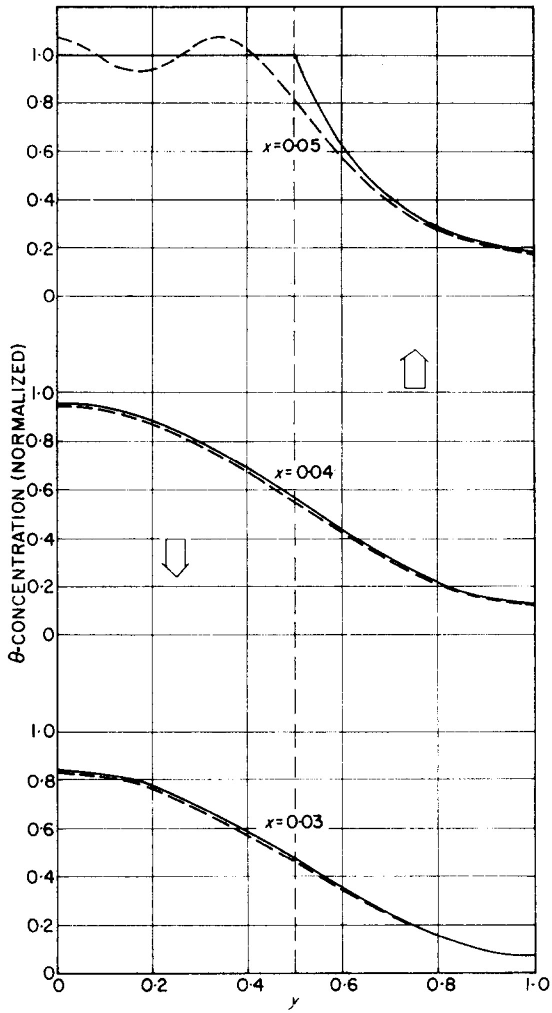

FIG. 6—*continued.*

finite difference methods; the points at $L = 0{\cdot}001$ come entirely from the finite difference method. Figure 9 also shows the mean Nusselt number for the counterflow heat exchanger problem discussed previously. Both the bulk concentration and the mean Nusselt numbers were calculated by numerical integration.

CONCLUDING REMARKS

The problem of $F = 0$, that is, of no net flow, cannot be handled by the series expansion technique as it has been developed here. This case introduces an indeterminate form in equations (18) and (19) since when $\lambda_0 = 0$, $Y_0 = 1$, and the denominators vanish. As noted by Bochef [2] this is a situation in which an eigenvalue whose order of multiplicity when regarded as a root of the characteristic equation (two in this case) is not equal to the number of linearly independent eigenfunctions corresponding to it (one in this case). This problem can, however, be solved by the finite difference method.

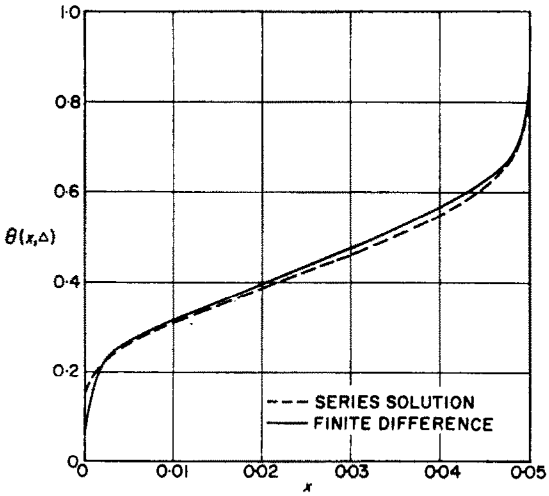

FIG. 7. Dimensionless interfacial concentration distribution for the case $L = 0{\cdot}05$, $N = 1{\cdot}33$.

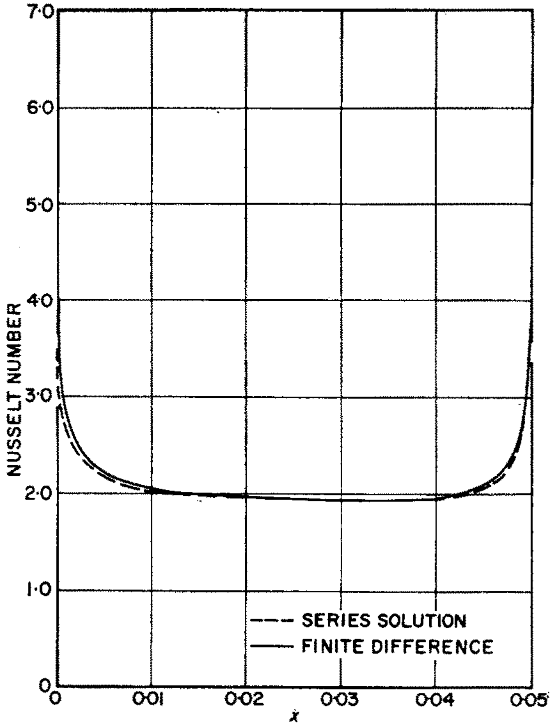

FIG. 8. Comparison of series and finite difference calculations of Nusselt number as a function of x for the case $L = 0{\cdot}05$, $N = 1{\cdot}33$.

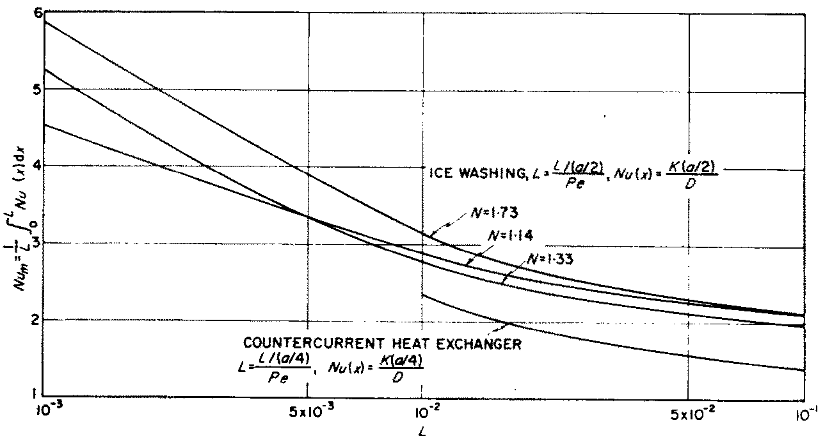

FIG. 9. Average Nusselt numbers for countercurrent heat or mass transfer.

ACKNOWLEDGEMENT

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REFERENCES

1. A. J. BARDUHN, U.S. Dept. of Interior, Office of Saline Water Conversion, Report No. 12, July (1958).
2. M. BOCHER, *Bull. Amer. Math. Soc.* **21**, 6 (1914).

3. E. L. INCE, *Ordinary Differential Equations*, Chapter 10. Longman-Green, London (1929).
4. C. J. KING, Lawrence Radiation Laboratory Report *UCRL*-11196, University of California, Berkeley, January (1964).
5. L. LAPIDUS, *Digital Computation for Chemical Engineers*. McGraw-Hill, New York (1962).
6. KOSAKU YOSIDA, *Lectures on Differential and Integral Equations*. Interscience, New York (1960).

Résumé—On expose une technique de développement orthogonal pour résoudre certains problèmes de transfert de chaleur ou de masse dans des systèmes à contre-courant et on l'applique à deux problèmes de transport idéalisés. La méthode de résolution est basée sur la théorie de Sturm et demande à la fois des valeurs propres positives et négatives pour être complète. Les nombres de Nusselt locaux et moyens sont donnés comme des fonctions des paramètres des deux systèmes.

Une méthode itérative de différences finies pour résoudre les mêmes problèmes est décrite et les résultats des deux méthodes sont comparés.

Zusammenfassung—Zur Lösung einiger Wärme- und Stoffübergangsprobleme in Gegenstromsystemen, wird eine Reihenentwicklung nach orthogonalen Funktionen hergeleitet und auf zwei idealisierte Übergangsprobleme angewandt. Die Lösungsmethode beruht auf der Sturm'schen Theorie und erfordert zur Vollständigkeit sowohl positive wie auch negative Eigenwerte. Lokale und mittlere Nusselt Zahlen werden als Funktionen der in den zwei Systemen vorkommenden Parameter angegeben. Zur Lösung der gleichen Probleme wird ein iteratives Differenzenverfahren beschrieben und die Ergebnisse beider Methoden werden miteinander verglichen.

Аннотация—Разработан метод ортогонального разложения для решения некоторых задач переноса тепла и массы в системах с противотоком; этот метод применен к двум идеализированным задачам переноса. Метод решения основан на теории Штурма и требует для полноты системы собственных функций как положительных, так и отрицательных собственных значений. Локальные и средние значения критерия Нуссельта представлены как функции параметров обеих систем.

Для решения этих же задач приводится итерационный разностный метод и дано сравнение результатов обоих методов.